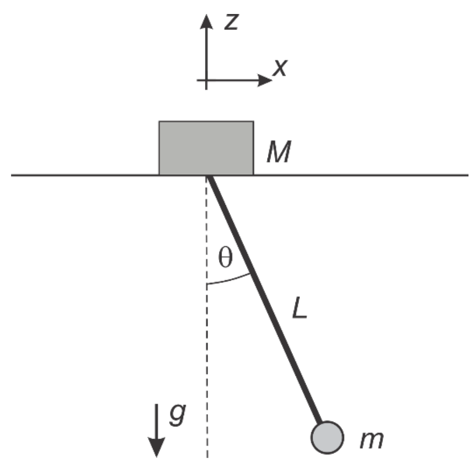
Ph.D. QUALIFYING EXAMINATION – PART A

Tuesday, January 10, 2023, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some students find useful the Schaum's outline, 'Mathematical Handbook of Formulas and Tables'.

A1. Block and pendulum



A block of mass M is located on a frictionless horizontal track. It can move freely in the x-direction. A point mass m is suspended from the block by a massless rod of length L. It can move in the xz-plane. The system is under the influence of a constant gravitational acceleration g.

a) Derive the Lagrangian of the system.

b) Approximate the Lagrangian for small oscillations around $\theta = 0$ and derive the equations of motion.

c) Determine the frequency of the small oscillations.

d) Discuss the behavior of the frequency in the limiting cases $M >> m$ and $M << m$.

A2. A current I is flowing in the $\hat{z}$ direction down a long wire of radius R. The current is distributed such that $I = a\,\rho^3$, where a is a constant and ρ is the perpendicular distance from the $\hat{z}$-axis which runs down the center of the wire.

a) Determine the magnetic field $\vec{B}$ inside and outside the long wire.

b) Sketch your results: $|B_\phi(\rho)| vs\ \rho$.

c) Determine the vector potential $\vec{A}$ inside and outside the long wire.

d) Sketch your results: $|A_\phi(\rho) vs\ \rho\,|$.

Recall: $\vec{B} = \vec{\nabla} \times \vec{A} = \hat{\rho}\left[\frac{1}{\rho}\frac{\partial A_z}{\partial \emptyset} - \frac{\partial A_\emptyset}{\partial z}\right] + \hat{\phi}\left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right] + \frac{\hat{z}}{\rho}\left[\frac{\partial(\rho A_z)}{\partial \rho} - \frac{\partial A_\rho}{\partial \emptyset}\right]$

A3. A Fermi gas of N particles with spin $\frac{1}{2}$ and mass m are placed inside a cube of linear size L with impenetrable walls. The temperature of the system is $T = 0$.

a) In terms of system parameters, derive an expression for the Fermi energy E_F of the gas.
b) Show that the average energy of the particles is $\frac{3}{5}E_F$.

A4. Consider the following thought experiment: a beam of particles with angular momentum $L = 1$ (regardless of the origin, either orbital or spin, in units of $\hbar$) is placed in a non-homogeneous magnetic field oriented along the x-axis. As you would (hopefully) expect, the beam of particles will separate into three components, which correspond to the eigenvalues of the L_x operator of $-1, 0$, and +1.

a. Case 1: You discard the -1 and +1 components, and only keep the zero'th component. Then you place it into the non-uniform magnetic field along the *z*-axis. Into how many components will the beam separate? Show your work.

b. Case 2: You discard the 0'th and the -1 components and only keep the $L_x=1$ component. Then you place it into the non-uniform magnetic field along the *z*-axis. What are the probabilities of finding the different (-1 , 0, and +1) eigenstates of L_z?

You might find useful the following matrix representations of the operators involved in the basis of the eigenstates of $\hat{L}^2, \hat{L}_z$:

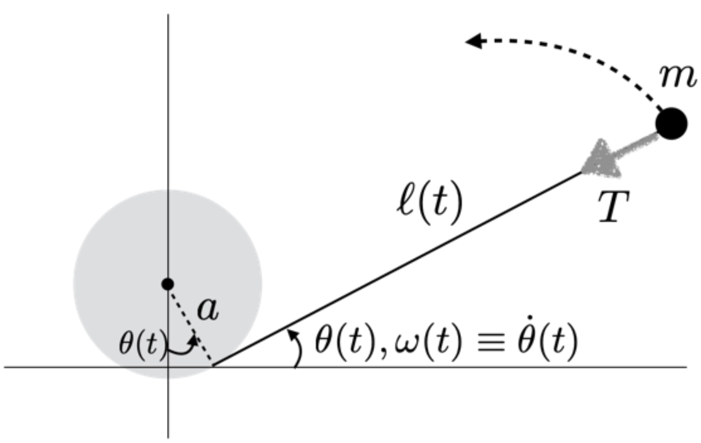
$$\hat{L}_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};\ \hat{L}_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix};\ \hat{L}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

A5. WRAPPING STRING

As shown in the figure below, a particle of mass m is moving in a horizontal plane perpendicular to a fixed cylinder with radius a. The particle is attached to a taut string that wraps around the cylinder, forcing the particle to spiral towards the cylinder. At a given time t, $\ell(t)$ is the length of the string not wrapped around the cylinder, and $\theta(t)$ is the angle of the string's tangent point to the cylinder measured from the initial position. Let $\ell(t = t_0) = 0$, and initial angular velocity of $\dot{\theta}(t = 0) = \omega_0$. A string tension T is the only force acting on the particle. You may use a, t_0, and ω_0 in your answers.

1) Find a constraint equation for ℓ and $\omega = d\theta/dt = \dot{\theta}$ from the fact that the string is wrapping around the cylinder.

2) Find an equation of motion for the angular momentum of the particle. Combining this with the constraint equation, find $\omega(t)$ and ℓ(t). Discuss if the angular momentum of the particle is conserved.



A6. Consider a system with three-levels $|g_1\rangle$, $|g_2\rangle$, and $|e\rangle$ described by the Hamiltonian

$$\hat{H} = \hbar\Omega_1|e\rangle\langle g_1| + \hbar\Omega_1^*|g_1\rangle\langle e| + \hbar\Omega_2|e\rangle\langle g_2| + \hbar\Omega_2^*|g_2\rangle\langle e| \ .$$

where Ω_1 and Ω_2 are complex coupling constants.

a) What are the eigenenergies and eigenstates for real $\Omega_1 = \Omega_2 = \Omega > 0$.
b) For the situation described in part (a), assume an initial state $|\psi(0)\rangle = |g_1\rangle$ and calculate the state $|\psi(t)\rangle$ at time t. Find the minimal time $T > 0$ for which $|\psi(T)\rangle = |g_2\rangle$.
c) For the general case with complex coupling parameters $\Omega_1$and Ω_2, show that there is an eigenstate with zero eigenenergy. Derive an expression for the normalized eigenstate of zero energy.
d) The result of problem (c) can be generalized for an n-level system with $n \geq 3$ described by the Hamiltonian

$$\hat{H} = \sum_{i=1}^{n-1} \hbar[\Omega_i|e\rangle\langle g_i| + \Omega_i^*|g_i\rangle\langle e|] \ ,$$

where the n levels are represented by the normalized kets $|g_i\rangle$ $(i = 1, \dots, n-1)$ and $|e\rangle$. Show that there are $(n-2)$ eigenstates of $\hat{H}$ with zero eigenenergy.

Hint: Use the ansatz $|\phi\rangle = \sum_{i=1}^{n-1} c_i\,|g_i\rangle + c_e|e\rangle$ and find the conditions for the coefficients c_i and c_e for which the state $|\phi\rangle$ is an eigenstate of the Hamiltonian with zero energy.

Ph.D. QUALIFYING EXAMINATION – PART B

Wednesday, January 11, 2023, 1:00 – 5:00 p.m.

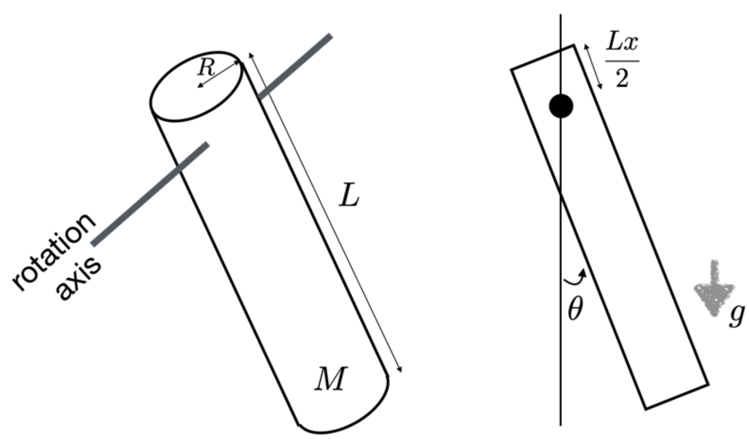
Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some students find useful the Schaum's outline', 'Mathematical Handbook of Formulas and Tables'.

B1. CYLINDER PENDULUM

Consider a uniform cylinder with length L, radius R, and mass M. The cylinder is pivoted at a fixed point whose distance from the top is $\frac{Lx}{2}$, where $(0 < x < 1)$, and it is allowed to rotate about the pivot point. The cylinder is free to rotate about the pivot axis due to gravity whose constant acceleration g is directed vertically downwards. You may use the given symbols and constants.

1) Find the cylinder's moment of inertia around the indicated rotation axis. Explicitly carry out the integrations needed to determine the moment of inertia for full credit.

2) Using the small angle approximation, $\theta \ll 1$, find the period of the cylinder pendulum. You may assume $L \gg R$. Also, find the value of x which minimizes the period of oscillation.



B2. Consider two point particles of mass m and charges $\pm Q$ connected by a massless rod of length $2a$. The rod is free to rotate in 3D. The Hamiltonian of the system is $\hat{H}_0 = \frac{\hat{L}^2}{2I}$, $I = 2ma^2$, where $\hat{L}$ is the angular momentum operator.

a. What are the quantum numbers that describes the stationary states of this system? What are the energy eigenvalues and what are their degeneracies? No derivations are required here.

b. A constant electric field, $\vec{E} = E\hat{e}_z$ is applied along the z-axis. What is the Hamiltonian $\hat{H}$ of the new systems? By considering the operators $\hat{L}^2, \hat{L}_z$ and $\hat{H}$, identify the quantum numbers suitable for labeling of the stationary states of the new Hamiltonian.

c. By considering E to be small (i.e. the energy contribution from the term containing the electric field is much smaller than the kinetic energy) determine the first order correction to the energy of the ground state.

d. Determine the second order correction to the energy of the ground state.

You might find the explicit expressions for a few spherical harmonics useful:

$$Y_0^0 = \frac{1}{2}\frac{1}{\sqrt{\pi}}$$

$$Y_1^1 = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{i\phi}sin\theta \text{ ; } Y_1^0 = \frac{1}{2}\sqrt{\frac{3}{\pi}}cos\theta \text{ ; } Y_1^{-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{-i\phi}sin\theta \text{ ;}$$

B3. The average power Φ emitted by the Sun of mass $M \simeq 2.0\times10^{30} kg$ is 380 yottawatts, (1 yotta = 10^{24}). A dust particle, which can be approximated as a uniform sphere of radius R and density ρ that absorbs all light incident on it, experiences forces of gravitational attraction and the radiative pressure. One can neglect all other interactions.

a) Derive an expression for R, in terms of system parameters and, if necessary, the fundamental constants if the dust particle is in equilibrium.

b) How does your answer depend on the distance from the Sun? Provide a physical explanation.

c) Which particles, with radii smaller or larger than your estimate, will be ejected out of the Solar System?

d) Provide a numerical estimate for R for the particles with $\rho = 2\times10^3 kg/m^3$.

B4. A sphere of homogeneous linear dielectric material is placed in an otherwise uniform electric field $\vec{E} = E_0\hat{z}$. The sphere has a dielectric constant κ .

a) Determine the electric potential $V(r,\theta)$ inside and outside the dielectric sphere.

b) Determine the electric field $\vec{E}(r,\theta)$ inside and outside the dielectric sphere.

c) Determine the polarization $\vec{P}(r,\theta)$ inside and outside the dielectric sphere.

Since the problem has azimuthal symmetry, the general solution to Laplace's equation for the potential is given as: $V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$, where $P_\ell(x)$ is a Legendre polynomial. A few of these are $P_0 = 1$, $P_1 = x$, $P_2 = (3x^2 - 1)/2$ and

$\int_{-1}^{1} P_\ell(x)\, P_m(x) dx = \frac{2}{2\ell+1}\,\delta_{\ell m}$.

B5. E&M:

Consider a charge Q, which is uniformly distributed over the surface of an insulating sphere of radius R. The sphere spins around an axis through its center (the z-axis) with an angular frequency ω.

a) What are the surface charge density σ (i.e., charge per area) and the surface current density $\vec{K}$ on the surface of the sphere?

b) What is the electric field inside and outside of the sphere?

c) Give an expression for the magnetic moment $\vec{\mu}$ of the system.

Hint: The magnetic moment of a circular current loop is proportional to the current times its area
.

d) Inside and outside of the sphere (i.e., for $r \neq R$) the generated magnetic field $\vec{B}$ can be expressed as the (negative) gradient of a scalar magnetic potential Φ_M (i.e., $\vec{B} = -\vec{\nabla}\Phi_M$). It can be shown that

$$\Phi_M = \begin{cases} A_i r \cos\theta & \text{for } \; r < R \\ A_o \dfrac{\cos\theta}{r^2} & \text{for } \; r > R \end{cases}$$

where A_i and A_o must fulfill the correct boundary conditions. Give and expression for the magnetic field inside and outside of the sphere.

B6. Cooling an ideal gas by effusion

A cubic box of linear size L contains a classical ideal gas consisting of N particles of mass m. The system is in equilibrium at a temperature T. A small hole of cross section A is made in one of the walls, allowing particles to escape.

a) Find the number of escaping particles per unit of time right after the hole is opened (in terms of m, A, N, L, and T). Also find the average energy per particle of the escaping particles. Compare it to the average energy of the particles in the box.

b) If the hole is sufficiently small, the gas inside the box remains in equilibrium (to a good approximation) during the effusion process. Using the equipartition theorem, $k_B T = (2E/3N)$, derive a differential equation for the time-dependence of the temperature in the box and solve it.

[Notes: (i) The system is macroscopic, i.e., N>>1 and A<<L^2. (ii) $\int_0^\infty dx\, x\, exp(-ax^2) = 1/(2a)$, $\int_0^\infty dx\, x^3\, exp(-ax^2) = 1/(2a^2)$, $\int_0^\infty dx\, x^5\, exp(-ax^2) = 1/a^3$]